Selected Solutions for Chapter 11: Hash Tables

Solution to Exercise 11.2-1

For each pair of keys k, l, where $k \neq l$, define the indicator random variable $X_{kl} = \text{I}\{h(k) = h(l)\}$. Since we assume simple uniform hashing, $\Pr\{X_{kl} = 1\} = \Pr\{h(k) = h(l)\} = 1/m$, and so $\text{E}[X_{kl}] = 1/m$.

Now define the random variable Y to be the total number of collisions, so that $Y = \sum_{k \neq l} X_{kl}$. The expected number of collisions is

$$\begin{aligned}
\text{E}[Y] &= \text{E}\left[\sum_{k \neq l} X_{kl}\right] \\
&= \sum_{k \neq l} \text{E}[X_{kl}] \qquad \text{(linearity of expectation)} \\
&= \binom{n}{2} \frac{1}{m} \\
&= \frac{n(n-1)}{2} \cdot \frac{1}{m} \\
&= \frac{n(n-1)}{2m} .
\end{aligned}$$

Solution to Exercise 11.2-4

The flag in each slot will indicate whether the slot is free.

- A free slot is in the free list, a doubly linked list of all free slots in the table. The slot thus contains two pointers.
- A used slot contains an element and a pointer (possibly NIL) to the next element that hashes to this slot. (Of course, that pointer points to another slot in the table.)

Operations

- ***Insertion:***
  - If the element hashes to a free slot, just remove the slot from the free list and store the element there (with a NIL pointer). The free list must be doubly linked in order for this deletion to run in $O(1)$ time.
  - If the element hashes to a used slot j, check whether the element x already there "belongs" there (its key also hashes to slot j).
    - If so, add the new element to the chain of elements in this slot. To do so, allocate a free slot (e.g., take the head of the free list) for the new element and put this new slot at the head of the list pointed to by the hashed-to slot (j).
    - If not, E is part of another slot's chain. Move it to a new slot by allocating one from the free list, copying the old slot's (j's) contents (element x and pointer) to the new slot, and updating the pointer in the slot that pointed to j to point to the new slot. Then insert the new element in the now-empty slot as usual.
    To update the pointer to j, it is necessary to find it by searching the chain of elements starting in the slot x hashes to.
- ***Deletion:*** Let j be the slot the element x to be deleted hashes to.
  - If x is the only element in j (j doesn't point to any other entries), just free the slot, returning it to the head of the free list.
  - If x is in j but there's a pointer to a chain of other elements, move the first pointed-to entry to slot j and free the slot it was in.
  - If x is found by following a pointer from j, just free x's slot and splice it out of the chain (i.e., update the slot that pointed to x to point to x's successor).
- ***Searching:*** Check the slot the key hashes to, and if that is not the desired element, follow the chain of pointers from the slot.

All the operations take expected $O(1)$ times for the same reason they do with the version in the book: The expected time to search the chains is $O(1 + \alpha)$ regardless of where the chains are stored, and the fact that all the elements are stored in the table means that $\alpha \leq 1$. If the free list were singly linked, then operations that involved removing an arbitrary slot from the free list would not run in $O(1)$ time.

Solution to Problem 11-2

a. A particular key is hashed to a particular slot with probability $1/n$. Suppose we select a specific set of k keys. The probability that these k keys are inserted into the slot in question and that all other keys are inserted elsewhere is

$$\left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k} .$$

Since there are $\binom{n}{k}$ ways to choose our k keys, we get

$$Q_k = \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k} \binom{n}{k} .$$

b. For $i = 1, 2, \ldots, n$, let X_i be a random variable denoting the number of keys that hash to slot i, and let A_i be the event that $X_i = k$, i.e., that exactly k keys hash to slot i. From part (a), we have $\Pr\{A\} = Q_k$. Then,

$$\begin{aligned}
P_k &= \Pr\{M = k\} \\
&= \Pr\left\{\left(\max_{1 \le i \le n} X_i\right) = k\right\} \\
&= \Pr\{\text{there exists } i \text{ such that } X_i = k \text{ and that } X_i \le k \text{ for } i = 1, 2, \ldots, n\} \\
&\le \Pr\{\text{there exists } i \text{ such that } X_i = k\} \\
&= \Pr\{A_1 \cup A_2 \cup \cdots \cup A_n\} \\
&\le \Pr\{A_1\} + \Pr\{A_2\} + \cdots + \Pr\{A_n\} \qquad \text{(by inequality (C.19))} \\
&= nQ_k .
\end{aligned}$$

c. We start by showing two facts. First, $1 - 1/n < 1$, which implies $(1 - 1/n)^{n-k} < 1$. Second, $n!/(n-k)! = n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) < n^k$. Using these facts, along with the simplification $k! > (k/e)^k$ of equation (3.18), we have

$$\begin{aligned}
Q_k &= \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k} \frac{n!}{k!(n-k)!} \\
&< \frac{n!}{n^k k!(n-k)!} && ((1 - 1/n)^{n-k} < 1) \\
&< \frac{1}{k!} && (n!/(n-k)! < n^k) \\
&< \frac{e^k}{k^k} && (k! > (k/e)^k) .
\end{aligned}$$

d. Notice that when $n = 2$, $\lg\lg n = 0$, so to be precise, we need to assume that $n \ge 3$.

In part (c), we showed that $Q_k < e^k/k^k$ for any k; in particular, this inequality holds for k_0. Thus, it suffices to show that $e^{k_0}/{k_0}^{k_0} < 1/n^3$ or, equivalently, that $n^3 < {k_0}^{k_0}/e^{k_0}$.

Taking logarithms of both sides gives an equivalent condition:

$$\begin{aligned}
3\lg n &< k_0(\lg k_0 - \lg e) \\
&= \frac{c \lg n}{\lg\lg n}(\lg c + \lg\lg n - \lg\lg\lg n - \lg e) .
\end{aligned}$$

Dividing both sides by $\lg n$ gives the condition

$$\begin{aligned}
3 &< \frac{c}{\lg\lg n}(\lg c + \lg\lg n - \lg\lg\lg n - \lg e) \\
&= c\left(1 + \frac{\lg c - \lg e}{\lg\lg n} - \frac{\lg\lg\lg n}{\lg\lg n}\right) .
\end{aligned}$$

Let x be the last expression in parentheses:

$$x = \left(1 + \frac{\lg c - \lg e}{\lg\lg n} - \frac{\lg\lg\lg n}{\lg\lg n}\right) .$$

We need to show that there exists a constant $c > 1$ such that $3 < cx$.

Noting that $\lim_{n\to\infty} x = 1$, we see that there exists n_0 such that $x \geq 1/2$ for all $n \geq n_0$. Thus, any constant $c > 6$ works for $n \geq n_0$.

We handle smaller values of n—in particular, $3 \leq n < n_0$—as follows. Since n is constrained to be an integer, there are a finite number of n in the range $3 \leq n < n_0$. We can evaluate the expression x for each such value of n and determine a value of c for which $3 < cx$ for all values of n. The final value of c that we use is the larger of

- 6, which works for all $n \geq n_0$, and
- $\max_{3 \leq n < n_0}\{c : 3 < cx\}$, i.e., the largest value of c that we chose for the range $3 \leq n < n_0$.

Thus, we have shown that $Q_{k_0} < 1/n^3$, as desired.

To see that $P_k < 1/n^2$ for $k \geq k_0$, we observe that by part (b), $P_k \leq nQ_k$ for all k. Choosing $k = k_0$ gives $P_{k_0} \leq nQ_{k_0} < n \cdot (1/n^3) = 1/n^2$. For $k > k_0$, we will show that we can pick the constant c such that $Q_k < 1/n^3$ for all $k \geq k_0$, and thus conclude that $P_k < 1/n^2$ for all $k \geq k_0$.

To pick c as required, we let c be large enough that $k_0 > 3 > e$. Then $e/k < 1$ for all $k \geq k_0$, and so e^k/k^k decreases as k increases. Thus,

$$\begin{aligned} Q_k &< e^k/k^k \\ &\leq e^{k_0}/k^{k_0} \\ &< 1/n^3 \end{aligned}$$

for $k \geq k_0$.

e. The expectation of M is

$$\begin{aligned} \mathrm{E}[M] &= \sum_{k=0}^{n} k \cdot \Pr\{M = k\} \\ &= \sum_{k=0}^{k_0} k \cdot \Pr\{M = k\} + \sum_{k=k_0+1}^{n} k \cdot \Pr\{M = k\} \\ &\leq \sum_{k=0}^{k_0} k_0 \cdot \Pr\{M = k\} + \sum_{k=k_0+1}^{n} n \cdot \Pr\{M = k\} \\ &\leq k_0 \sum_{k=0}^{k_0} \Pr\{M = k\} + n \sum_{k=k_0+1}^{n} \Pr\{M = k\} \\ &= k_0 \cdot \Pr\{M \leq k_0\} + n \cdot \Pr\{M > k_0\} , \end{aligned}$$

which is what we needed to show, since $k_0 = c \lg n / \lg\lg n$.

To show that $\mathrm{E}[M] = O(\lg n / \lg\lg n)$, note that $\Pr\{M \leq k_0\} \leq 1$ and

$$
\begin{aligned}
\Pr\{M > k_0\} &= \sum_{k=k_0+1}^{n} \Pr\{M = k\} \\
&= \sum_{k=k_0+1}^{n} P_k \\
&< \sum_{k=k_0+1}^{n} 1/n^2 \qquad \text{(by part (d))} \\
&< n \cdot (1/n^2) \\
&= 1/n \; .
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
\mathrm{E}[M] &\le k_0 \cdot 1 + n \cdot (1/n) \\
&= k_0 + 1 \\
&= O(\lg n / \lg\lg n) \; .
\end{aligned}
$$